Three temperatures distribution function model

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The three temperatures model has been introduced for simulating an anistropic tail in the electron momentum distribution function driven by the quasi-electrostatic wave at the Lower Hybrid frequency in tokamak. It is a very crude model which was used to characterize the fast electron bremsstrahlung during these non-inductive regimes, which extends up to high photon energies corresponding to hard X rays.

With the emergence of refined models based on the use of Fokker-Planck solvers, the three temperatures model is almost obsolete, and its application is restricted to benchmark non-thermal bremsstrahlung codes designed for analysing corresponding diagnostics.

Let define the 3-D distribution function $f(\psi, p, \xi)$, where ψ is the poloidal magnetic flux coordinate (radial position), p the momentum value and $\xi = p_{\parallel}/p$ the pitch-angle relative to the local magnetic field direction. Here, f is considered to be homogeneous on a magnetic flux surface, and the trapped electron population is neglected. According to the three temperatures model,

$$f(\psi, p, \xi) = \lambda_M f_M(\psi, p, \xi) + \lambda_{3T} f_{3T}(\psi, p, \xi) \quad (1)$$

where $f_M(\psi, p, \xi)$ is the thermal bulk, and $f_{3T}(\psi, p, \xi)$ is the non-thermal part. By definition, both f_M and f_{3T} are normalized to unity, so that f automatically satisfies this condition. The relative fraction of fast electrons which is roughly given by the ratio λ_{3T}/λ_M is usually very small, of the order of 1×10^{-3}.

1 The Maxwellian distribution function

The thermal distribution function is expressed in its general relativistic form

$$f_M(\psi, p, \xi) \propto \alpha_M \exp\left[-\frac{p^2}{(1+\gamma)\Theta}\right] = \alpha_M \exp\left[-\frac{\gamma - 1}{\Theta}\right] \quad (2)$$

where $\Theta(\psi) = T_e(\psi)/m_e c^2$ is the ratio of the local electron temperature $T_e(\psi)$ to the electron rest mass energy $m_e c^2$ while the relativistic Lorentz correction factor is defined as $\gamma^2 = p^2 + 1$. Here, ξ is the cosine of the pitch angle. By symmetry, the Maxwellian distribution function is independant of ξ. By definition, f_M is normalized to unity in the interval $p \in [0, p_{min}[$, where p_{min} is a given upper cut-off limit for the Maxwellian. The parameter α_M is then given by the integral

$$4\pi\alpha_M \int_0^{p_{min}} p^2 \exp\left[-\frac{\gamma-1}{\Theta}\right] dp = 1 \tag{3}$$

Recalling that $\gamma d\gamma = pdp$,

$$\alpha_M = \frac{1}{4\pi \exp\left(\Theta^{-1}\right) \Xi_p\left(\Theta^{-1}, p_{min}\right)} \tag{4}$$

where

$$\Xi_p\left(\Theta^{-1}, p_{min}\right) = \int_1^{\sqrt{1+p_{min}^2}} \gamma\sqrt{\gamma^2-1} \exp\left[-\gamma\Theta^{-1}\right] d\gamma \tag{5}$$

By integrating by parts

$$\begin{aligned}\Xi_p\left(\Theta^{-1}, p_{min}\right) &= \frac{p_{min}^3}{3} \exp\left[-\sqrt{1+p_{min}^2}\Theta^{-1}\right] \\ &+ \frac{\Theta^{-1}}{3} \int_1^{\sqrt{1+p_{min}^2}} \left(\gamma^2-1\right)^{3/2} \exp\left[-\gamma\Theta^{-1}\right] d\gamma \qquad (6)\end{aligned}$$

and in the limit $p_{min} \to +\infty$,

$$\Xi_p\left(\Theta^{-1}\right) = \frac{\Theta^{-1}}{3} \int_1^{\infty} \left(\gamma^2-1\right)^{3/2} \exp\left[-\gamma\Theta^{-1}\right] d\gamma \tag{7}$$

$$= \frac{4}{3}\frac{(3/2)!}{\sqrt{\pi}} \frac{K_2\left(\Theta^{-1}\right)}{\Theta^{-1}} \tag{8}$$

where $K_2(z)$ is the modified Bessel function of order 2.

For $\Theta \ll 1$, using the large argument asymptotic development

$$K_2\left(\Theta^{-1}\right) \approx \sqrt{\pi/2}\sqrt{\Theta}\exp(-\Theta^{-1})\left(1 + 15\Theta/8 + O\left(\Theta^{-2}\right)\right) \tag{9}$$

the usual expression keeping the first in the expansion

$$f_M(\psi, p, \xi) \simeq \frac{1}{[2\pi\Theta]^{3/2}} \exp\left[-\frac{\gamma-1}{\Theta}\right] \tag{10}$$

$$= \frac{1}{[2\pi\Theta]^{3/2}} \exp\left[-\frac{p^2}{(1+\gamma)\Theta}\right] \tag{11}$$

is well recovered and

$$\alpha_M \simeq [2\pi\Theta]^{-3/2} \tag{12}$$

In the LUKE code, the momentum p here expressed in relativistic units $m_e c$ is normalized to the thermal reference value $p_{th}^{\dagger} = m_e v_{th}^{\dagger}$,

$$\overline{p} = p/p_{th}^{\dagger} \tag{13}$$

and consequently

$$p_{th}^{\dagger}/m_e c \approx v_{th}^{\dagger}/c = \beta_{th}^{\dagger} \tag{14}$$

since thermal electrons are only weakly relativistic. The well known relativistic Lorentz correction factor γ is then simply given by the relation

$$\gamma = \sqrt{p^2 + 1} = \sqrt{\overline{p}^2 \beta_{th}^{\dagger 2} + 1} \tag{15}$$

and in the non-relativistic limit, i.e. the condition $\overline{p}^2 \beta_{th}^{\dagger 2} \ll 1$ or $\gamma \approx 1$ holds.

Since in relativistic units, the total energy is linked to the relativistic momentum by the expression,

$$(E_c + 1)^2 = \gamma \tag{16}$$

it is straightforward to express the kinetic energy E_c as a function of $\overline{p}$ in units of electron rest mass energy $m_e c^2$

$$E_c = m_e c^2 \left(\sqrt{\overline{p}^2 \beta_{th}^{\dagger 2} + 1} - 1 \right) \tag{17}$$

Finally, concerning the normalization of the electron velocity v, one has

$$\begin{aligned} -1 \ll v/c &= p/\left(\gamma m_e c\right) && (18)\\ &= \overline{p} p_{th}^{\dagger}/\left(\gamma m_e c\right) && (19)\\ &= \overline{p} \beta_{th}^{\dagger}/\gamma && (20) \end{aligned}$$

and using $\overline{v} = v/v_{th}^{\dagger}$, it comes

$$\overline{v} v_{th}^{\dagger}/c = \overline{p} \beta_{th}^{\dagger}/\gamma \tag{21}$$

or

$$\overline{v} = \overline{p}/\gamma \tag{22}$$

with

$$v_{th}^{\dagger}/c = \beta_{th}^{\dagger} \tag{23}$$

Consequently, the non-relativistic expression of f_M is only valid when $\gamma \simeq 1$, or in an equivalent form

$$\overline{p}^2 \beta_{th}^{\dagger 2} \ll 1 \tag{24}$$

Since $\overline{p}$ may be as large as 30 in numerical calculations, in order to correctly describe momentum dynamics of the fastest electrons, it results that

$$\beta_{th}^{\dagger 2} = \Theta^{\dagger} \ll 1/900 \tag{25}$$

or

$$T_e^{\dagger} \ll \frac{511}{900} \approx 0.57 keV \tag{26}$$

since $\beta_{th}^{\dagger} = \sqrt{T_e^{\dagger}/m_e c^2}$ and $\Theta^{\dagger} = T_e^{\dagger}/m_e c^2$. Therefore, in a thermonuclear plasma, one must always consider the relativistic form of the Maxwellian distribution function. In addition, considering the asymptotic limit of K_2 for large arguments (9), the condition is

$$\frac{15}{8}\Theta^{\dagger} \ll 1 \tag{27}$$

or

$$T_e^{\dagger} \ll 272keV$$

which is always satisfied, since $T_e^{\dagger}$ never exceeds a few ten keV in tokamak plasmas. This means that the approximate formulation of f is fully valid.

In normalized units,

$$f_M(\psi, \overline{p}, \xi) = \frac{1}{[2\pi\Theta(\psi)]^{3/2}} \exp\left[-\frac{\overline{p}^2 \beta_{th}^{\dagger 2}}{(1+\gamma)\overline{T}_e(\psi)\beta_{th}^{\dagger 2}}\right] \tag{28}$$

$$= \frac{1}{\left[2\pi\overline{T}_e(\psi)\beta_{th}^{\dagger 2}\right]^{3/2}} \exp\left[-\frac{\overline{p}^2}{(1+\gamma)\overline{T}_e(\psi)}\right] \tag{29}$$

using the relation $\Theta = \overline{\Theta}\Theta^{\dagger}$, with $\overline{\Theta} = \overline{T}_e$. Then, it turns out that

$$f_M(\psi, \overline{p}, \xi) = \frac{1}{\beta_{th}^{\dagger 3}} \overline{f}_M(\psi, \overline{p}, \xi) = \frac{1}{p_{th}^{\dagger 3}} \overline{f}_M(\psi, \overline{p}, \xi) \tag{30}$$

since $p_{th}^{\dagger} = \beta_{th}^{\dagger}$, with

$$\overline{f}_M(\psi, \overline{p}, \xi) \approx \frac{1}{\left[2\pi\overline{T}_e(\psi)\right]^{3/2}} \exp\left[-\frac{\overline{p}^2}{(1+\gamma)\overline{T}_e(\psi)}\right] \tag{31}$$

and the modified normalisation coefficient becomes

$$\overline{\alpha}_M \simeq \left[2\pi\overline{T}_e(\psi)\right]^{-3/2} \tag{32}$$

The Maxwellian distribution function be expressed in an alternative form, useful for calculating interpolation between full and half-grids,

$$\overline{f}_M(\psi, \overline{p}, \xi) \approx \frac{1}{\left[2\pi\overline{T}_e(\psi)\right]^{3/2}} \exp\left[-\frac{\gamma - 1}{\overline{T}_e(\psi)\beta_{th}^{\dagger 2}}\right] \tag{33}$$

One can then easily cross-check that

$$2\pi \int_{-1}^{+1} d\xi_0 \int_0^{+\infty} p^2 f_M(\psi, p, \xi)\, dp = \frac{2\pi p_{th}^{\dagger 3}}{\beta_{th}^{\dagger 3}} \int_{-1}^{+1} d\xi_0 \int_0^{+\infty} \overline{p}^2 \overline{f}_M(\psi, \overline{p}, \xi)\, d\overline{p} = 1 \tag{34}$$

is well recovered.

2 The non-maxwellian tail

The tail part of the distribution function can be expressed in the simple form

$$\begin{aligned} f_{3T}\left(\psi,p,\xi\right) &= \alpha_{3T} H\left(p_{max}-p\right) H\left(p-p_{min}\right) \times \\ &\left\{ \exp\left[-\frac{p_{\parallel}^2}{2\Theta_{\parallel f}\left(\psi\right)}\right] \exp\left[-\frac{p_{\perp}^2}{2\Theta_{\perp}\left(\psi\right)}\right] H\left(p_{\parallel}\right) \right. \\ &\left. + \exp\left[-\frac{p_{\parallel}^2}{2\Theta_{\parallel b}\left(\psi\right)}\right] \exp\left[-\frac{p_{\perp}^2}{2\Theta_{\perp}\left(\psi\right)}\right] \left(1-H\left(p_{\parallel}\right)\right) \right\} \end{aligned} \quad (35)$$

where $H(x)$ is the usual Heaviside function while $\Theta_{\parallel f}$, $\Theta_{\parallel b}$ and $\Theta_{\perp}$ are respectively the parallel forward, backward and perpendicular pseudo-temperatures normalized to $m_e c^2$. They are considered as pseudo-temperatures, since they do not correspond to a thermodynamic equilibrium for which the electron temperature T_e has a true physical meaning, i.e. the mean thermal velocity. They are introduced as simple parameters to characterize the lack of symmetry around a given direction. Here, the parallel component is expressed as

$$p_{\parallel} = p\xi = \mathbf{p} \cdot \hat{b} \quad (36)$$

where $\hat{b}$ is the unitary vector on the magnetic field line. Conversely, $p_{\perp}$ is defined as $p^2 = p_{\parallel}^2 + p_{\perp}^2$. The distribution function f_{3T} is also normalized to unity. Let p_{min} being the momentum value corresponding to the intersection of f_{3T} with f_M, and p_{max}, the upper limit above which it is zero.

$$\int_{-\infty}^{+\infty} dp_{\parallel} \int_{0}^{+\infty} f_{3T}\left(\psi,p,\xi\right) p_{\perp} dp_{\perp} = 1 \quad (37)$$

The presence of the Heaviside functions makes the calculation of the coefficient α_{3T} non trivial. In the limits $lim_{p_{min}\to 0} f_{3T}$ and $lim_{p_{max}\to +\infty} f_{3T}$, the parallel and perpendicular dynamics are decoupled, and an analytical expression may be derived. Indeed, In this case,

$$\begin{aligned} \alpha_{3T} \left(\int_{0}^{+\infty} \exp\left[-\frac{p_{\parallel}^2}{2\Theta_{\parallel f}}\right] dp_{\parallel} + \int_{-\infty}^{0} \exp\left[-\frac{p_{\parallel}^2}{2\Theta_{\parallel b}}\right] dp_{\parallel} \right) & \\ \times \int_{0}^{+\infty} \exp\left[-\frac{p_{\perp}^2}{2\Theta_{\perp}}\right] 2\pi p_{\perp} dp_{\perp} &= 1 \end{aligned} \quad (38)$$

and a simple integration gives

$$\int_{0}^{+\infty} \exp\left[-\frac{p_{\perp}^2}{2\Theta_{\perp}}\right] 2\pi p_{\perp} dp_{\perp} = 2\pi\Theta_{\perp} \quad (39)$$

while

$$\int_{0}^{+\infty} \exp\left[-\frac{p_{\parallel}^2}{2\Theta_{\parallel f}}\right] dp_{\parallel} = \sqrt{\frac{\pi}{2}}\sqrt{\Theta_{\parallel f}} \quad (40)$$

Therefore, it is straightforward to find that

$$\alpha_{3T} = \frac{1}{\sqrt{2\pi^3}} \frac{1}{\Theta_\perp \left(\sqrt{\Theta_{\parallel f}} + \sqrt{\Theta_{\parallel b}}\right)} \tag{41}$$

and for the Maxwellian case $\Theta_{\parallel f} = \Theta_{\parallel b} = \Theta_\perp = \Theta$, one recovers well the relation

$$\alpha_{3T} = \alpha_M = (2\pi\Theta)^{-3/2} \tag{42}$$

Considering the reference to the thermal value $p_{th}^\dagger$, defining $p_\parallel = \overline{p}_\parallel p_{th}^\dagger$ and $p_\perp = \overline{p}_\perp p_{th}^\dagger$, as well as for the temperatures, one finds

$$\begin{aligned} f_{3T}(\psi,\overline{p},\xi) &= \overline{\alpha}_{3T} H(\overline{p}_{max} - \overline{p}) H(\overline{p} - \overline{p}_{min}) \times \\ &\left\{ \exp\left[-\frac{\overline{p}_\parallel^2}{2\overline{T}_{\parallel f}(\psi)}\right] \exp\left[-\frac{\overline{p}_\perp^2}{2\overline{T}_\perp(\psi)}\right] H\left(\overline{p}_\parallel\right) \right. \\ &\left. + \exp\left[-\frac{\overline{p}_\parallel^2}{2\overline{T}_{\parallel b}(\psi)}\right] \exp\left[-\frac{\overline{p}_\perp^2}{2\overline{T}_\perp(\psi)}\right] \left(1 - H\left(\overline{p}_\parallel\right)\right) \right\} \end{aligned} \tag{43}$$

using the identity for the Heaviside function

$$H(ax+b) = H\left(x + \frac{b}{a}\right) H(a) + H\left(-x - \frac{b}{a}\right) H(-a) \tag{44}$$

and the fact that $p_{th}^\dagger > 0$. The modified normalized coefficient is immediately

$$\overline{\alpha}_{3T} = \frac{1}{\sqrt{2\pi^3}} \frac{1}{\overline{T}_\perp \left(\sqrt{\overline{T}_{\parallel f}} + \sqrt{\overline{T}_{\parallel b}}\right)} \tag{45}$$

3 Calculation the distribution

For a given value of the coefficient λ_{3T} which represents approximately the fraction of suprathermal electrons above the value $\overline{p}_{min}$ the normalisation of the distribution function $\overline{f}$ is calculated with the constraint

$$\lambda_M \left(1 - \delta_M (\overline{p}_{min})\right) + \lambda_{3T} \left(1 - \delta_{3T} (\overline{p}_{min})\right) = 1 \tag{46}$$

so that

$$\lambda_M = \frac{1 - \lambda_{3T}\left(1 - \delta_{3T}(\overline{p}_{min})\right)}{1 - \delta_M(\overline{p}_{min})}$$

where δ_M and δ_{3T} are by definition small corrections, i.e. $0 \leq \delta_M \ll 1$ and $0 \leq \delta_{3T} \ll 1$, provided $\lambda_{3T} \ll 1$. Indeed, in this case, $\overline{p}_{min} \gg p_{th}^\dagger$, while the contribution of the region $\overline{p} \leq \overline{p}_{min}$ to the deviation of the normalisation coefficient is almost negligible. The choice of λ_{3T} and $\overline{p}_{min}$ must be set by the condition

$$\delta_{3T}(\overline{p}_{min}) \geq 1 - \frac{1}{\lambda_{3T}}$$

since $\lambda_M \geq 0$.